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ACHILLES AND THE TORTOISE

By MAX BLACK

1. SUPPOSE Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. In order to win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on, without end. Achilles never catches the tortoise, because the tortoise always holds a lead, however small.

This is approximately the form in which the so-called "Achilles" paradox has come down to us. Aristotle, who is our primary source for this and the other paradoxes attributed to Zeno, summarises the argument as follows: "In a race the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead" (Physics, 239^b).[1]

2. It would be a waste of time to prove, by independent argument, that Achilles will pass the tortoise. Everybody knows this already, and the puzzle arises because the conclusion of Zeno's argument is known to be absurd. We must try to find out, if we can, exactly what mistake is committed in this argument.[2]

[1] Aristotle's solution seems to be based upon a distinction between two meanings of 'infinite'—(i) as meaning "infinite in extent", (ii) as meaning "infinitely divisible". "For there are two senses in which length and time and generally anything continuous are called 'infinite': they are called so either in respect of divisibility or in respect of their extremities. So while a thing in a finite time cannot come in contact with things quantitatively infinite, it can come in contact with things infinite in respect of divisibility; for in this sense the time itself is also infinite..." (*Physics*, 233^a). This type of answer has been popular (cf. *e.g.* J. S. Mill, *System of Logic*, 5th ed., 389–390). Several writers object that infinite divisibility of the line implies its actually having an infinite number of elements—and so leaves the puzzle unresolved. But see H. R. King, "Aristotle and the paradoxes of Zeno", *Journal of Philosophy* 46 (1949), 657–670.

For references to the vast literature on this and the other arguments of Zeno, see F. Cajori, "The history of Zeno's arguments on motion", *American Mathematical Monthly* 22 (1915), 1–6, 39–47, 77–82, 109–115, 143–149, 179–186, 253–258, 292–297.

[2] It has sometimes been held (*e.g.* by Paul Tannery in *Revue Philosophique* 20 (1885) that Zeno's arguments were sound. "Tannery's explanation of the four arguments, particularly of the 'Arrow' and 'Stade' raises these paradoxes from childish arguments to arguments with conclusions which follow with compelling force . . . it exhibits Zeno as a logician of the first rank" (Cajori, *op. cit.*, 6).

Cf. Russell's remark that the arguments of Zeno "are not, however, on any view, mere foolish quibbles: they are serious arguments, raising difficulties which it has taken two thousand years to answer, and which even now are fatal to the teachings of most philosophers" (*Our Knowledge of the External World* (1926), 175).

3. A plausible answer that has been repeatedly offered[1] takes the line that "this paradox of Zeno is based upon a mathematical fallacy" (A. N. Whitehead, *Process and Reality* (1929), 107).

Consider the lengths that Achilles has to cover, according to our version of the paradox. They are, successively, a hundred yards, ten yards, one yard, a tenth of a yard, and so on. So the total number of yards he must travel in order to catch the tortoise is

$$100 + 10 + 1 + 1/10 + \ldots$$

This is a convergent geometrical series whose sum can be expressed in decimal notion as 111. $\dot{1}$, that is to say exactly $111\frac{1}{9}$. When Achilles has run this number of yards, he will be dead level with his competitor; and at any time thereafter he will be actually ahead.

A similar argument applies to the time needed for Achilles to catch the tortoise. If we suppose that Achilles can run a hundred yards in ten seconds, the number of seconds he needs in order to catch up is

$$10 + 1 + 1/10 + 1/100 + \ldots$$

This, too, is a convergent geometrical series, whose sum is expressed in decimal notation 11. $\dot{1}$, that is to say exactly $11\frac{1}{9}$. This, as we should expect, is one tenth of the number we previously obtained.

We can check the calculation without using infinite series at all. The relative velocity with which Achilles overtakes the tortoise is nine yards per second. Now the number of seconds needed to cancel the initial gap of a hundred yards at a relative velocity of pursuit of nine yards per second is 100 divided by 9 or $11\frac{1}{9}$. This is exactly the number we previously obtained by summing the geometrical series representing the times needed by Achilles. During this time, moreover, since Achilles is actually travelling at ten yards per second, the actual distance he travels is $10 \times 11\frac{1}{9}$, or $111\frac{1}{9}$, as before. Thus we have confirmed our first calculation by an argument not involving the summation of infinite series.

4. According to this type of solution, the fallacy in Zeno's argument is due to the use of the words "never" and "always". Given the premise that "the pursuer must first reach the point whence the pursued started," it does *not* follow, as alleged, that the quickest runner "never" overtakes the slower: Achilles

[1] In addition to the reference to Whitehead, see for instance Descartes (letter to Clerselier, Adam and Tannery, ed. of *Works* 4, 445–447), and Peirce (Collected Papers, 6.177–6.182). Peirce says "... this silly little catch presents no difficulty at all to a mind adequately trained in mathematics and in logic ..." (6.177).

does catch the tortoise at some time—that is to say at a time exactly $11\frac{1}{9}$ seconds from the start. It is wrong to say that the tortoise is " always " in front : there is a place—a place exactly $111\frac{1}{9}$ yards from Achilles' starting point—where the two are dead level. Our calculations have showed this, and Zeno failed to see that only a finite time and finite space are needed for the infinite series of steps that Achilles is called upon to make.

5. This kind of mathematical solution has behind it the authority of Descartes and Peirce and Whitehead[1]—to mention no lesser names—yet I cannot see that it goes to the heart of the matter. It tells us, correctly, when and where Achilles and the tortoise will meet, *if* they meet ; but it fails to show that Zeno was wrong in claiming they *could not* meet.

Let us be clear about what is meant by the assertion that the sum of the infinite series.

$$100 + 10 + 1 + 1/10 + 1/100 + \ldots$$

is $111\frac{1}{9}$. It does not mean, as the naive might suppose, that mathematicians have succeeded in adding together an infinite number of terms. As Frege pointed out in a similar connection,[2] this remarkable feat would require an infinite supply of paper, an infinite quantity of ink, and an infinite amount of time. If we had to add all the terms together, we could never prove that the series had a finite sum. To say that the sum of the series is $111\frac{1}{9}$ is to say that if enough terms of the series are taken, the difference between the sum of that *finite number* of terms and the number $111\frac{1}{9}$ becomes, and stays, as small as we please. (Or to put it another way : Let n be any number less than $111\frac{1}{9}$. We can always find a finite number of terms of the series whose sum will be less than $111\frac{1}{9}$ but greater than n).

Since this is all that is meant by saying that the infinite series has a sum, it follows that the " summation " of all the terms of an infinite series is not the same thing as the summation of a finite set of numbers. In one case we can get the answers by working out a finite number of additions ; in the other case we *must* " perform a limit operation ", that is to say, prove that there is a number whose difference from the sum of the initial members of the series can be made to remain as small as we please.

6. Now let us apply this to the race. The series of distances traversed by Achilles is convergent. This means that if Achilles takes enough steps whose sizes are given by the series 100 yards, 10 yards, 1 yard, 1/10 yard, etc. the distance *still to go* to the meeting point eventually becomes, and stays, as small as we

[1] See the last footnote.

[2] *Grundgesetze der Arithmetik* 2 (1903) ; §124. Or see my translation in *Philosophical Review* 59 (1950), 332.

please. After the first step he still has $11\frac{1}{9}$ yards to go; after the second, only $1\frac{1}{9}$ yard; after the third, no more than $\frac{1}{9}$ yard; and so on. The distance still to go is reduced by ten at each move.

But the distance, however much reduced, still remains to be covered; and after each step there are infinitely many steps still to be taken. The logical difficulty is that Achilles seems called upon to perform *an infinite series of tasks*; and it does not help to be told that the tasks become easier and easier, or need progressively less and less time in the doing. Achilles may get nearer to the place and time of his rendezvous, but his task remains just as hard, for he still has to perform what seems to be logically impossible. It is just as hard to draw a very small square circle as it is to draw an enormous one: we might say both tasks are infinitely hard. The logical difficulty is not in the extent of the distance Achilles has to cover but in the apparent impossibility of his travelling any distance whatsoever. I think Zeno had enough mathematical knowledge to understand that if Achilles could run $111\frac{1}{9}$ yards—that is to say, keep going for $11\frac{1}{9}$ seconds—he would indeed have caught the tortoise. The difficulty is to understand how Achilles could arrive anywhere at all without first having performed an infinite series of acts.

7. The nature of the difficulty is made plainer by a second argument of Zeno, known as the "Dichotomy" which, according to Aristotle, is "the same in principle" (*Physics*, 239^b). In order to get from one point to another, Achilles must first reach a third point midway between the two; similarly, in order to reach this third point he must first reach a fourth point; to reach this point he must first reach another point; and so on, without end. To reach *any* point, he must first reach a nearer one. So, in order to get moving, Achilles must already have performed an infinite series of acts—must, as it were, have travelled along the series of points from the infinitely distant and *open* "end".[1] This is an even more astounding feat than the one he accomplishes in winning the race against the tortoise.

The two arguments are complementary: the "Achilles" shows that the runner cannot reach any place, even if he gets started; while the "Dichotomy" shows that he cannot get started, *i.e.* cannot leave any place he has reached.

8. Mathematicians have sometimes said that the difficulty of conceiving the performance of an infinite series of tasks is factitious. All it shows, they say, is the weakness of human

[1] This, at any rate, is the usual interpretation, though I cannot see that Aristotle was thinking of anything more than an argument resembling the "Achilles" in all respects except that of the ratio in which the distance is divided. For the contrary view see, for instance, Sir Thomas Heath, *Mathematics in Aristotle* (1949), 135–6.

imagination and the folly of the attempt to make a mental image of mathematical relationships.[1] The line really does have infinitely many points, and there is no logical impediment to Achilles' making an infinite number of steps in a finite time. I will try to show that this way of thinking about the race is untenable.

9. I am going to argue that the expression, " infinite series of acts ", is self-contradictory, and that failure to see this arises from confusing a series of acts with a series of numbers generated by some mathematical law. (By an " act " I mean something marked off from its surroundings by having a definite beginning and end.)

In order to establish this by means of an illustration I shall try to make plain some of the absurd consequences of talking about " counting an infinite number of marbles ". And in order to do this I shall find it convenient to talk about counting an infinite number of marbles as if I supposed it was sensible to talk in this way. But I want it to be understood all the time that I do not think it sensible to talk in this way, and that my aim in so talking is to show how absurd this way of talking is. Counting may seem a very special kind of " act " to choose, but I hope to be able to show that the same considerations will apply to an infinite series of any kinds of acts.

10. Suppose we want to find out the number of things in a given collection, presumably identified by some description. Unless the things are mathematical or logical entities it will be impossible to deduce the size of the collection from the description alone ; and somebody will have to do the work of taking a census. Of course he can do this without having any idea of how large the collection will turn out to be : his instructions may simply take the form, " Start counting and keep on until there is nothing left in the collection to count ". This implies that there will be a point at which there will be " nothing left to count ", so that the census-taker will then know his task to have been completed.

Now suppose we can know that the collection is infinite. If, knowing this, we were to say " Start counting and continue until there is nothing left to count " we should be practicing a deception. For our census-taker would be led to suppose that sooner or later there would be nothing left to count, while all

[1] " La perception sensible n'embrasse que le fini ; l'imagination atteint encore les infiniment grands et les infiniment petits, tant qu'ils restent finis ; mais elle n'atteint ni l'infini, limite des infiniment grands, ni le *zéro*, limite des infiniment petits : ces deux etats extrémes de la grandeur sont de pures idées, accessibles à la seule raison " (L. Couturat, *De l'infini mathématique* (1896), 562).

the time we would know this supposition to be false. An old recipe for catching guinea pigs is to put salt on their tails. Since they have no tails, this is no recipe at all. To avoid deception we should have said, in the case of the infinite collection, " Start counting and *never* stop ". This should be enough to tell an intelligent census-taker that the collection is infinite, so that there is no sense in trying to count it.

If somebody says to me " Count all the blades of grass in Hyde Park" I might retort "It's too difficult; I haven't enough time ". But if some cosmic bully were to say " Here is an infinite collection; go ahead and count it," only logical confusion could lead me to mutter " Too difficult; not enough time ". The trouble is that, no matter what I do, the result of all my work will not and cannot count as compliance with the instructions. If somebody commands me to obey a certain " instruction ", and is then obliging enough to add that nothing that I can do will count as compliance with that instruction, only confusion could lead me to suppose that any task had been set.

11. Some writers, however, have said that the difficulty of counting an infinite collection is just a matter of *lack of time*.[1] If only we could count faster and faster, the whole job could be done in a finite time ; there would still never be a time at which we were ending, but there would be a time at which we already would have ended the count. It is not necessary to finish counting ; it is sufficient that the counting shall have been finished.

Very well. Since the task is too much for human capacity, let us imagine a machine that can do it. Let us suppose that upon our left a narrow tray stretches into the distance as far as the most powerful telescope can follow; and that this tray or slot is full of marbles. Here, at the middle, where the line of marbles begins, there stands a kind of mechanical scoop ; and to the right, a second, but empty tray, stretching away into the distance beyond the farthest reach of vision. Now the machine is started. During the first minute of its operation, it seizes a marble and transfers it to the empty tray ; then it rests a minute. In the next half-minute the machine seizes a second marble on the left, transfers it, and rests half-a-minute. The third marble is moved in a quarter of a minute, with a corresponding pause ; the next in one eighth of a minute ; and so until the movements are so fast that all we can see is a grey blur. But at the end of exactly

[1] " Quand vous dites qu'une collection infinie ne pourra jamais être numérotée tout entière, il ne s'agit pas là d'une impossibilité intrinsique et logique, mais d'une impossibilité pratique et matérielle : c'est tout simplement une question de temps " (L. Couturat, *op. cit.* (1896), 462).

four minutes the machine comes to a halt, and we now see that the left-hand tray that was full seems to be empty, while the right-hand tray that was empty seems full of marbles.

Let us call this an *infinity machine*. And since it is the first of several to be described let us give it the name "Alpha".

12. I hope nobody will object that the wear and tear on such a machine would be too severe; or that it would be too hard to construct! We are dealing with the logical coherence of ideas, not with the practicability of mechanical devices. If we can conceive of such a machine without contradiction, that will be enough; and believers in the "actual infinite" will have been vindicated.

13. An obvious difficulty in conceiving of an infinity machine is this. How are we supposed to know that there are infinitely many marbles in the left-hand tray at the outset? Or, for that matter, that there are infinitely many on the right when the machine has stopped? Everything we can observe of Alpha's operations (and no matter how much we slow it down!) is consistent with there having been involved only a very large, though still finite, number of marbles.

14. Now there is a simple and instructive way of making certain that the machine shall have infinitely many marbles to count. Let there be only *one* marble in the left-hand tray to begin with, and let some device always return *that same marble* while the machine is resting. Let us give the name 'Beta' to a machine that works in this way. From the standpoint of the machine, as it were, the task has not changed. The difficulty of performance remains exactly the same whether the task, as in Alpha's case, is to transfer an infinite series of qualitatively similar but different marbles; or whether the task, as in Beta's case, is constantly to transfer the *same* marble that is immediately returned to its original position. Imagine Alpha and Beta both at work side by side on their respective tasks: every time the one moves, so does the other; if one succeeds in its task, so must the other; and if it is impossible for either to succeed, it is impossible for *each*.

15. The introduction of our second machine, Beta, shows clearly that the infinite count really is impossible. For the single marble is always returned, and each move of the machine accomplished nothing. A man given the task of filling three holes by means of two pegs can always fill the third hole by transferring one of the pegs; but this automatically creates another empty place, and it won't help in the least to run through this futile series of operations faster and faster. (We

don't get any nearer to the end of the rainbow by running faster). Now our machine, Beta, is in just this predicament: the very act of transferring the marble from left to right immediately causes it to be returned again; the operation is self-defeating and it is logically impossible for its end to be achieved. Now if this is true for Beta, it must be true also for Alpha, as we have already seen.

16. When Hercules tried to cut off the heads of Hydra, two heads immediately grew where one had been removed. It is rumoured that the affair has been incorrectly reported: Zeus, the all powerful, took pity on Hercules and eased his labor. It was decreed that only *one* head should replace the head that had been cut off and that Hercules should have the magical power to slash faster and faster in geometrical progression. If this is what really happened, had Hercules any cause to be grateful? Not a bit. Since the head that was sliced off immediately grew back again, Hercules was getting nowhere, and might just as well have kept his sword in its scabbard.

17. Somebody may still be inclined to say that nevertheless when the machine Beta finally comes to rest (at the end of the four minutes of its operation) the single marble might after all be found in the right-hand tray, and this, if it happened, would *prove* that the machine's task had been accomplished. However, it is easy to show that this suggestion will not work.

I said, before, that "some device" always restored the marble to its original position in the left-hand tray. Now the most natural device to use for this purpose is another machine —Gamma, say—working like Beta but *from right to left*. Let it be arranged that no sooner does Beta move the marble from left to right than Gamma moves it back again. The successive working periods and pauses of Gamma are then equal in length to those of Beta, except that Gamma is working while Beta is resting, and vice versa. The task of Gamma, moreover, is exactly parallel to that of Beta, that is, to transfer the marble an infinite number of times from one side to the other. If the result of the whole four minutes' operation by the first machine is to transfer the marble from left to right, the result of the whole four minutes' operation by the second machine must be to transfer the marble from right to left. But there is only one marble and it must end somewhere! Hence neither machine can accomplish its task, and our description of the infinity machines involves a contradiction.

18. These consideration show, if I am not mistaken, that the outcome of the infinite machine's work is independent of

what the machine is supposed to have done antecedently. The marble might end up on the right, on the left, or nowhere. When Hercules ended his slashing, Zeus had to decide whether the head should still be in position or whether, after all, Hercules' strenuous efforts to do the impossible should be rewarded.

Hercules might have argued that every time a head appeared, he had cut it off, so no head ought to remain; but the Hydra could have retorted, with equal force, that after a head had been removed another had always appeared in its place, so a head ought to remain in position. The two contentions cancel one another and neither would provide a ground for Zeus' decision.

Even Zeus, however, could not abrogate the continuity of space and motion; and this, if I am not mistaken, is the source of the contradiction in our description of the machine Beta. The motion of the marble is represented, graphically, by a curve with an infinite number of oscillations, the rapidity of the oscillations increasing constantly as approach is made to the time at which the machine comes to rest. Now to say that motion is continuous is to deny that any real motion can be represented by a curve of this character. Yet every machine that performed an infinite series of acts in a finite time would have to include a part that oscillated " infinitely fast ", as it were, in this impossible fashion. For the beginning of every spatio-temporal act is marked by a change in the velocity or in some other magnitude characterizing the agent.

19. It might be thought that the waiting-intervals in the operations of the three infinity machines so far described have been essential to the argument. And it might be objected that the steps Achilles takes are performed consecutively and without intervening pauses. I will now show that the pauses are not essential.

Consider for this purpose two machines, Delta and Epsilon, say, that begin to work with a single marble each, but in opposite directions. Let Delta start with the marble *a* and Epsilon with the marble *b*. Now suppose the following sequence of operations: while Delta transfers marble *a* from left to right in one minute, Epsilon transfers marble *b* from right to left; then Delta moves *b* from left to right in half a minute while Epsilon returns *a* from right to left during the same time; and so on, indefinitely, with each operation taking half the time of its predecessor. During the time that either machine is transporting a marble, its partner is placing the other marble in position for the next move. Once again, the total tasks of Delta and Epilson are exactly parallel: if the first is to succeed, both marbles must

end on the right, but if the second is to succeed, both must end on the left. Hence neither can succeed, and there is a contradiction in our description of the machines.

20. Nor will it help to have a machine—Phi, say—, transferring marbles that become progressively smaller in geometrical progression.[1] For, by an argument already used, we can suppose that while Phi is performing its operations, one of the machines already described is going through its paces at the same rates and at the same times. If Phi could complete its task, Alpha, Beta, Gamma, Delta and Epsilon would have to be able to complete their respective tasks. And we have already seen that this is not possible. The size of the successive tasks has nothing to do with the logical impossibility of performing an infinite series of operations. Indeed it should be clear by this time that the logical possibility of the existence of any one of the machines depends upon the logical possibility of the existence of all of them or, indeed, of any machine that could count an infinite number of objects. If the idea of the existence of any one of them is self-contradictory, the same must be true for each of them. The various descriptions of these different hypothetical devices simply make it easier for us to see that one and all are logically impossible. And though a good deal more needs to be said about this, I hope I have said enough to show why I think this notion of counting an infinite collection is self-contradictory.

21. If we now reconsider for a moment the arguments that have been used in connection with our six infinity machines, we can easily see that no use was made of the respects in which counting differs from any other series of acts. Counting differs from other series of acts by the conventional assignment of numerals to each stage of the count, and it differs in other respects, too. But every series of acts is like counting in requiring the successive doing of things, each having a beginning and end in space or time. And this is all that was used or needed in our arguments. Since our arguments in no way depended upon the specific peculiarities of counting they would apply, as I said at the outset, to any infinite series of acts.

22. And now let us return to Achilles. If it really were necessary for him to perform an infinite number of *acts* or, as Aristotle says " to pass over or severally to come in contact with infinite things " (*Physics*, 233[a]), it would indeed be logically impossible for him to pass the tortoise. But all the things he

[1] An alternative arrangement would be to have three similar machines constantly circulating three marbles.

[1] Somebody might say that if the marble moved by Beta eventually shrunk to nothing there would be no problem about its final location !

really does are finite in number; a finite number of steps, heart beats, deep breaths, cries of defiance, and so on. The track on which he runs has a finite number of pebbles, grains of earth, and blades of grass,[1] each of which in turn has a finite, though enormous number of atoms. For all of these are things that have a beginning and end in space or time. But if anybody says we must imagine that the atoms themselves occupy space and so are divisible " in thought ", he is no longer talking about spatio-temporal things. To divide a thing " in thought " is merely to halve the numerical interval which we have assigned to it. Or else it is to suppose what is in fact physically impossible beyond a certain point, the actual separation of the physical thing into discrete parts. We can of course choose to say that we shall represent a distance by a numerical interval, and that every part of that numerical interval shall also count as respresenting a distance; then it will be true a priori that there are infinitely many " distances ". But the class of what will then be called " distances " will be a series of pairs of numbers, not an infinite series of spatio-temporal things. The infinity of this series is then a feature of one way in which we find it useful to *describe* the physical reality; to suppose that therefore Achilles has to *do* an infinite number of things would be as absurd as to suppose that because I can attach two numbers to an egg I must make some special effort to hold its halves together.

23. To summarise: I have tried to show that the popular mathematical refutation of Zeno's paradoxes will not do, because it simply assumes that Achilles can perform an infinite series of acts. By using the illustration of what would be involved in counting an infinite number of marbles, I have tried to show that the notion of an infinite series of acts is self-contradictory. For any material thing, whether machine or person, that set out to do an infinite number of acts would be committed to performing a motion that was discontinuous and therefore impossible. But Achilles is not called upon to do the logically impossible; the illusion that he must do so is created by our failure to hold separate the finite number of real things that the runner has to accomplish and the infinite series of numbers by which we describe what he actually does. We create the illusion of the infinite tasks by the kind of mathematics that we use to describe space, time, and motion.

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[1] Cf. Peirce: " I do not think that if each pebble were broken into a million pieces the difficulty of getting over the road would necessarily have been increased; and I don't see why it should if one of these millions—or all of them—had been multiplied into an infinity " (*Op. cit.*, 6.182).

ON ARGUMENTS FOR REAL UNIVERSALS

By A. C. LLOYD

1. THE most widely accepted arguments for the reality of universals have always been from the nature of discourse. It is only such arguments that will be considered in this article. Its purpose is to suggest that arguments from *ordinary* discourse are probably invalid, but that, according to current expositions, the foundations of *mathematical* discourse give better hope for the realist. But I am almost entirely concerned with the first, or destructive, suggestion. The second will be treated only allusively; and it will be apparent that I am open to refutation there not merely by the future of mathematical logic but by a better knowledge of its present.

2. "Nominalism" and "realism" are vague labels. And what kinds of terms are claimed to stand for abstract universals is a matter of choice. *E.g.* one can, if one wishes, dispense in logic with classes and keep relations, or equally dispense with relations and keep classes. But the traditional question of real *versus* post rem universals still survives in the form, Can a language be constructed which would contain all statements agreed to be true, but would imply none asserting the existence of anything but individuals? I leave 'individuals' undefined. There is no trick about them; and the question can be thought of in the ordinary way as the question whether classes or concepts, relations, etc., are removeable by Ockham's razor. An affirmative answer will be called nominalism, a negative realism.

3. Ockham's razor is taken as a postulate.

4. All positive "demonstrations" of realism (from the nature of ordinary discourse) are probably reducible to "unum nomen unum nominatum". This is, in a sense, a description or slogan rather than a demonstration. And if it reflects an argument this contains a *petitio*. For when I write or say, "This is red" and "That is red", and assume that there is one name (type) 'red', as there are two shapes or noises (tokens) 'red', I am already assuming a real universal, viz. the name.

5. Realists therefore fall back on negative demonstrations, *i.e.* prove nominalism logically untenable. I shall take some widely believed proofs of this, and suggest that each is invalid.

6. First the "in respect of" argument. An infinite regress in nominalism is claimed as follows. Nominalists say that "*a* is red" ≡ "*a* resembles *s*", where *s* is some standard red

object, say this penny stamp. But they have to add, " in respect of colour ", otherwise the meaning might be taken as " a is rectangular ". And we now have a universal, colour. To avoid this they can say instead that " *a* is red " ≡ " *a* resembles *s* and *t* ", *t* being, say, a red counter. But this might mean " *a* is flat " ; and we shall have no end to the number of objects that must be named.

But why will there be no end ? We can add, " . . . and *u* " (something red and rough) or " . . . but not *v* " (something flat and blue), and " . . . and not *w* " (something orange and rough) to rule out " *a* is warm-coloured ".

The realist's objection implies a negative answer to the question (Q) which runs, in his language, as follows. Can the instances (bundles) of simple universals be arranged in pairs whose members either resemble each other or are distinguishable in respect of only one simple universal ? For the nominalist is in effect treating universals as elements of classes, and requiring that every element may be specified by the intersection of a finite number of these classes ; and with a finite number of elements it can be shewn to follow that every element will be specifiable by the intersection either of two classes or of one class and the inverse of another. (This does not, however, mean that to specify any given universal, it is necessary to find one of the pairs of classes which would specify it, for it may also be specifiable by the intersection of a greater number of classes.)

Now (i) no evidence is forthcoming that the answer to (Q) is No ; and (ii) the question (though queer enough) seems to be an empirical one, so that the objection cannot be a *logical* refutation of nominalism.

For suppose it is retorted that (Q) is meaningless, since we can never know whether we have discovered all the simple properties of an object. To answer this we must take note of the two sorts of possibility.

(*a*) If " distinguishable " refers to physical (or psychological) possibility, (Q) can stand (as an empirical question), provided we take " can be arranged " to refer to *the same kind of possibility*. It is then irrelevant that a later discovery of another property may upset the classification. *E.g.*, suppose Greek artists made no distinction of warm and cold colours, their omission of *w* in our example would not invalidate the analysis of " This is red ". Modern people would have a different analysis ; but that is only because the notion of " standard object " is a relative one.

(*b*) If on the other hand " distinguishable " refers to logical (or metaphysical) possibility, *then so does " can be arranged "*.

If, further, it is claimed that there is now no conceivable proof of a positive answer, the claim is an *ignoratio elenchi*. For it would only refute our argument if this had been intended to demonstrate nominalism. But the argument was intended only to refute a certain refutation of nominalism; for that it was sufficient to rule out proofs of a negative answer to (Q), and these *will* be ruled out—along with proofs of a positive answer—by our opponent's claim. In fact the claim is neutral and harmless, because it draws attention only to the impossibility of completely analysing any empirical concept.

7. "Resemblance" argument. Nominalists define "a is red" as "a resembles *s*", where *s* is some standard red object. But to define "resembles" they have to say, for example

$$aRs \equiv (a\ R\ s)\ R_1\ (s\ R\ t).$$

But if (i) in the definiens the Rs in brackets are not the same as the R outside brackets, they then have to define the latter. This gives them, for example,

$$(aRs)\ R_1\ (sRt) \equiv [(aRs)\ R_1\ (sRt)]\ R_2\ [(sRt)\ R_1\ (tRu)]$$

and the same difficulty arises over the meaning of R_2. Thus they have an infinite regress. If (ii) at any stage they say one R is the same as another R, they have admitted one real universal, viz. resemblance.

The substance of the reply to this argument is that, if there is an infinite regress, it is not a vicious one. Let us see what was really happening. We were asked to analyse the proposition "*a* is like *s*". This we did by saying

$$aRs \equiv aRs \text{ is like } sRt,$$

where R is a relation of likeness which makes complexes out of a and *s* and out of *s* and t. We were then asked whether the 'like' was the same as the 'R'. Now this is so ill framed a question that we answer truthfully either Yes and No according to its meaning. If it means, Are the symbols 'like' and 'R' equivalent? the answer is of necessity Yes. For otherwise we should not have been giving an analysis. But we have not "admitted a real universal"; we have only *used* a general word, 'like'. If it is objected that, on the contrary, the question referred not to symbols, but what they stood for, the answer is that symbols "standing for" things which are not symbols is an elaborate metaphor. There is no real (in rebus) relation "standing for" like the relation of touching. The realist can only deny this by begging the whole question in his favour. He can now, however, rightly point out that likeness must be a real relation, for it is used in the analysis of a simple, sensible fact, such as that this is red. But the only way that he can get us

to consider real relations as opposed to symbols is to get us *not to analyse the proposition "a is like s", i.e. to state a dictionary definition of 'like', but to cause him to understand or shew him the meaning of the expression "this is like that"*. In doing so we shall draw his attention to some actual cases of likeness, and people do not ask whether certain actual objects are the same object; if they did, we should of necessity answer No; for otherwise there would not have been more than one there.

The only way in which the infinite regress would arise is by having a confused notion which of the two tasks is being undertaken. If we are giving a dictionary definition of 'like', we do *not*, as we did previously, repeat the word 'like', but we use a synonym such as 'similar to'. Of course if you want 'similar to' defined and then 'resembles' and so on, we shall eventually run out of synonyms. But there is nothing vicious here. Dictionary defining is circular, because dictionaries contain a finite number of words.[1]

8. A second "in respect of" argument. It may be said[2] that the nominalist arguments in 6. and 7. are wrong, for resemblance is *not* a real relation. Anything resembles anything in some way; so that it appears as a sort of ubiquitous and therefore meaningless predicate. "x resembles y" is an incomplete statement, for it always means "... in respect of P" (some property). This objection could be held to lie outside the scope of this article, since it seems to appeal to normal usage or to an intuition. In fact one can deny the necessity of its premises by saying that 'resembles' is not an incomplete symbol when it means 'is indistinguishable from',[3] and that this is the meaning relevant to simple properties, which is all we need be concerned with. It can be admitted that indistinguishability may often be a matter of subjective choice (just as the simplicity of properties often is). And the identity of indiscernibles—if we wanted the principle—would not, I think, entail realism, because 'indistinguishable' need not entail 'indiscernible'. For we can require the second, without requiring the first, to be transitive.[4]

9. The "recognition" argument. We often recognise that something is red without recognising a similarity with something else. This is irrelevant because we are not analysing "y recog-

[1] The reply in 7. could probably be put more rigorously in terms of a confusion between use and mention or between levels of language. But rigour has in this case been deliberately sacrificed for the hope of persuasiveness.

[2] See S. Hampshire *Scepticism and meaning* in *Philosophy* XXV (1950), pp. 237–44. He is not however arguing for realism.

[3] Mr. Hampshire admits this as a possibility (*loc. cit.* 238, n.1.).

[4] I omit here any nominalist theory which drops resemblance altogether—*e.g.* one which says that "a is red" ≡ (very roughly) "a is associated with the word 'red'".

nises that x is red", but "x is red". (A "verificationist" will need care here, but no more.)

10. "Degrees of resemblance" argument. Resemblance implies degrees. Sensible qualities may always be described in terms of resemblance because there it is always possible for some object to resemble another in a greater or less degree. But this will not do when we come to many relations, specially intransitive ones. *E.g.* it would not be false, but meaningless, to say that one man was the grandfather of another in a greater degree than someone else. Formally, "x is grandfather of y" cannot mean "xRy resembles vRw" where vRw is a standard complex containing the relation grandfather of; for if m resembles n, there can always be another object o, than which it resembles n to a greater degree; but this will not hold where m resembles n in respect of being a grandfather.

This objection is met (α) as in 8, by substituting indistinguishability for resemblance. One may also argue (β) that it is doubtful whether there are any relations which have no degrees, provided the relations are simple (*e.g.* sense data). If it is contradictory to have degrees, the relation is a *well defined symbol*. *E.g.* the word 'straight' as a simple term of first intention admits degrees, while as a mathematical term it does not.

11. I conclude that realism has a poor case in the logic of every day language. Defined generously however it has a much better one in mathematical logic. The paradoxes, such as Burali-Forti's (greatest ordinal number) and Russell's (class of all classes) shewed the inadequacy of traditional or intuitive notions of classes and concepts. But Russell's nominalist logic of incomplete symbols notoriously left propositional functions in place of real classes, and at least two difficulties seem to remain. For it makes no odds whether they are to be found in metalogic instead of logic.

12. First, propositional functions. They can hardly be dealt with by extreme nominalism, *i.e.* as classes of sentences. For, unlike the analysis of "this is red", that of "x is red" seems to involve an infinite number of sentences, and this is surely a contradiction if sentences are real things. If they are classes of propositions, these can perhaps be dealt with in psychology. But this has objections.

13. Secondly, pure mathematics. This is perhaps more important because the problem is capable of something like rigorous formulation. *E.g.* the theory of types (thought necessary for theory of classes) *plus* the theory of hierarchy of functions required a so-called Axiom of Reducibility (necessary for the

definition of real numbers). But this automatically re-introduces either classes or (what has other objections) an infinite number of functions. Modern Set Theory generally contains axioms asserting the existence of classes. All this of course soon leads into technicalities of very great difficulty. But they cannot be considered unimportant by those who still consider important the less technical arguments from discourse.

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PERCEPTUAL CLAIMS AND CERTAINTIES

By M. J. BAKER

AMONG recent philosophical writings there have been several attacks upon Phenomenalism and kindred doctrines. At the same time a view has developed which asserts that although many thinkers have supposed otherwise, the questions to which these doctrines seek to provide answers are not genuine, and should not have been asked. Thus it is thought to have been shown not only that these doctrines are false accounts of perceptual experience, but that no doctrines of this kind are required for theory of knowledge. I shall consider here not the doctrines and the attacks which have been made upon them, but a point concerning the accompanying suggestion that the questions which led to the formulation of the doctrines are not genuine.

Suppose I have a visual experience and at once or shortly afterwards say, sincerely, and in a language which I have learned to use, "I see a yellow ball". At the time when I make the statement I have no reason to suppose that I am mistaken; but I may be. For saying that I see a yellow ball entails that many other experiences are obtainable: more of the ball could be seen by moving about; it could be touched; if it were thrown into the air it would presently fall to the ground; etc. Failure to obtain some of these experiences might lead me to withdraw the statement, for at the time when I make it and at least for some time afterwards it lacks either logical or empirical certainty: it makes only a claim. I shall not consider here whether such a statement ever does acquire empirical certainty. If all that it entails is found to obtain for some considerable time after it is made and then failures occur, we should not readily withdraw the statement, we might find ourselves to be in a situation in which we did not know what to say. But whatever may be the status of such a statement after it has passed successfully many tests, nevertheless if at the time of a perception or shortly afterwards I report it by using a perception-verb with a material-object term as its object then my statement makes only a claim, which may have to be withdrawn. This is not to say that doubts are entertained about the correctness of the claim; it is to point out that at the time of the perception and at least for some time afterwards a material-object statement reporting it is of such a nature that it can only claim to be true; it may be mistaken.

Just as I do not want here to consider the status of such statements after they have passed successfully many tests, so I do not want to consider a claim often made that as there are some correct uses of perceptual material-object statements this implies that such statements are sometimes true. This controversial contention need not concern us now. I shall concentrate attention upon the case of having a perception and at the same time, or shortly afterwards, sincerely making a statement, in a language which I know how to use, such as "I see a yellow ball." In other words I shall consider what is involved in sincere first-person singular perceptual material-object statements.

Suppose I look in a certain direction and say " I see a yellow ball ", and follow this by saying " No ! What I see is an orange". This I should describe as being mistaken in my first statement and correcting it with my second. For to make a claim is to admit the possibility of being mistaken, and to admit the possibility of being mistaken is also to admit the possibility of correcting the mistake. What I wish to do is to draw attention to the fact that these sincere first-person singular material-object claim-making statements admit only one form of correction. If I say mistakenly " I see a yellow ball " then a correction statement can have the form only of substituting some other term for " yellow ball ". This is determined by the fact that the " I see " of this statement cannot be mistaken. I cannot offer as a correction a statement which substitutes another term for " I ". For between someone-else's seeing and my own seeing there is no possibility of mistake. Similarly I cannot offer as a correction a statement which substitutes some other term for " see ". For between my seeing and touching, or other mode of perceiving, there is no possibility of mistake. It would not make sense, as a correction statement, to say " I thought that I saw a yellow ball but in fact I touched it ". Of " I see a yellow ball ", with the qualifications given above, there is only one possible form of correction statement.

It may be contended that although I cannot be mistaken between someone-else's seeing and my own, or between my seeing and other modes of perception yet I may be mistaken between perceiving and imagining. It may be contended that " I imagined that I saw a yellow ball " could correct " I saw a yellow ball ". But I think that this contention is untenable unless the first statement is regarded as a substitution for a correction statement of the form " I saw . . ." For consider what is involved in imagining. Is it a form of perceptual experience ? If not, then it cannot be mistaken for such an

experience. If it is, then it is either a mode of perception which is additional to those of seeing, hearing, touching, smelling, and tasting, or it is a particular form of these modes. In the first case it could not be mistaken for one of the other modes of perception; while in the second its significance will be that of a mark indicating a distinction in the objects of perception. Thus, if imagining that I see is not a form of seeing, that is, is not a visual experience, then it cannot be mistaken for one; if it is a form of seeing, that is, a visual experience, then the significance of the distinction between imagining that I see and seeing is that of a mark indicating a distinction in objects of visual experience. Such a distinction will not affect the correctness of a statement of the form " I see ".

It may be contended that even if I cannot be mistaken between cases of seeing and imagining yet we use the verb " to see " in many senses, and I may be mistaken concerning its sense in any particular case of its use. But I think that the considerations stated above apply with equal force to this contention. In " I see a yellow ball " an essential element of the meaning of " see " is its reference to the occurrence of a visual experience. In " I see the point of your argument " there is no such reference. However many senses of " see " there may be, there is a clear distinction between those which indicate the occurrence of visual experiences and those which do not. Between these two classes of senses there is no possibility of mistake, and any differences within the class indicating the occurrence of visual experiences have significance as marks denoting differences in the nature of what is seen and do not affect the correctness of a statement of the form " I see ".

The " I see " of " I see a yellow ball ", with the qualifications already given, cannot be mistaken, and it is important to notice that this indubitableness is not that of logical necessity. " 'I perceive,' said sincerely by someone who has mastered the language being used, cannot be mistaken " need not be a logically necessary statement in order to be indubitable: the certainty of this statement arises from the inconceivability that " I perceive ", with the qualifications given above, could be mistaken; not from the linguistic rules which govern the use of the phrase. A statement which has a certainty determined by the rules for its use can always be mistaken when used sincerely by someone who has mastered the language: if found mistaken it is then withdrawn in order to preserve the certainty prescribed by the rules. To say that a statement cannot be mistaken when used sincerely by someone who has mastered the

language, is to draw attention to the fact that it has a certainty independent of the rules governing its use. Linguistic rules guarantee the certainty of some hypothetical statements, but the indubitableness to which I refer occurs in a class of statements which are categorical.

To say that a statement is mistaken is to claim to know (in some sense of " know ") what it would be like for the statement not to be mistaken : just as saying that something is wrong implies contrasting it with what is right. So to say that " I see a yellow ball " is mistaken is to imply that a correction statement could be made, and, as already noticed, this correction can have the form only of substituting some other term for "yellow ball". Suppose that I substitute " orange " for " yellow ball " then this may be found the implied correction. But if the substitution is made when the mistake is noticed, or shortly afterwards, then " I see an orange " can only claim to be the correction : it may be found to be a second mistaken statement ; the correction may be difficult to find. For all the time I make substitutions by employing material-object terms, I make claim-making statements which may be found to be mistaken : I may pass indefinitely to and fro between " yellow ball ", " orange ", " coloured piece of soap ", " large ripe peach ", " ripe yellow apple ", etc. But although such a bewildering situation is possible, the normal one is that in which I soon discover the required amendment.

But irrespective of whether a claim-making statement is found to be mistaken or correct there is the fact that the " I perceive " of a perceptual statement, with the qualifications given above, cannot be mistaken. Is this fact of any importance for epistemology ? I think it is. For perception-verbs are transitive : the apparently intransitive uses of them are elliptical. If I say " I see distinctly " I imply that what I do see is seen distinctly. The importance of noting that perception-verbs are transitive is that this transitivity involves that if at any time it is certain that I see then it is also certain that I see something. An indubitable statement could be made at this time having the form " I see . . . " This indubitable statement cannot be identical with a claim-making statement—irrespective of whether this is mistaken or correct—for these statements, at least while they remain at the claim-making level, are uncertain. Yet the indubitable statement could be made at the time when a sincere first-person singular perceptual claim-making statement is made, for it is involved in the making of such a statement. Whether such a statement is mistaken or correct is a matter which is established

by the application of tests to the claims of the statement. But as the "I perceive" of these statements is indubitable they involve in the very making of them indubitable statements of the form "I perceive . . .": for I cannot perceive without perceiving something. These latter statements do not make claims and their certainty does not await establishment through the application of tests.

It is not my aim in this discussion to attempt to say what sort of term would fill the blank in these statements: my object is to draw attention to the fact that indubitable statements are involved in the making of perceptual claims.

It has been contended[1] that the situation to which I might refer by saying "I thought that I saw a yellow ball but what I saw was an orange" is to be described as one in which an unusual view of an orange was mistaken for a typical view of a yellow ball, and that if there was a perceptual certainty then it was that an orange was seen. Stated in the terminology employed here, this view contends that the mistake which occurred was that the object of the correction statement looked as if it were an object of the kind mentioned in the mistaken statement. But this contention would lead to regarding the indubitable statements to which I am drawing attention as identical with correct material-object statements, and this discussion has tried to show that such identification is not possible. We are often advised not to waste time and effort searching for certainties when none are to be found. In the world in which we live we can only make claims and confirm or reject them. I have, however, tried to show that one particular search for indubitable statements is not a search for certainties when none is to be found, but an attempt to reveal the certainties involved in some perceptual claims.

It may be contended that these indubitable statements are those which follow analytically from material-object statements. Thus when I see a yellow ball, an orange or other similar object, I may say with certainty "I see something yellow". But the certainty of these statements is that of logical necessity, and their truth is logically dependent upon the establishment of the truth of material-object statements. Until the truth of a statement such as "I see a yellow ball" is established then the certainty of "I see something yellow" is to be found only in the hypothetical statement "If I see a yellow ball then I see something yellow". But the indubitable statements with which I am concerned have a certainty other than that of logical

[1] See Chapter 7, *The Concept of Mind*, by Prof. G. Ryle.

necessity, and they are logically independent of material-object statements.

So there are questions to ask about the indubitable statements involved in sincere first-person singular perceptual claim-making statements and about their relationship to claim-making statements ; whether these are of the indisputed or the mistaken or correction types. And this situation is unaffected by arguments suggesting that claims may be later converted into empirical certainties, or that the correct making of claims implies that what is claimed is sometimes true. Both these contentions may be true ; yet it is also true that there is a difference between making a claim and being certain, and whenever a sincere first-person singular perceptual claim is made there is at that time a perceptual certainty. Phenomenalism and kindred doctrines have sought to answer questions of this kind. If these doctrines are unacceptable—and I think they are—then other answers should be found. This may be of no practical importance : what we need to do in our everyday lives is to make claims and confirm or reject them. For daily life, therefore, these questions may be ignored ; but for theory of knowledge they await answers.

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CORRECTION

AMENDING THE VERIFICATION PRINCIPLE—BY ROBERT BROWN AND JOHN WATLING

ANALYSIS, Vol. 11, p. 88, l. 33, p. 89, lines 18 and 29: "... then it must not contain any component whose deletion leaves ..." *should read*, "... then it must contain only components whose deletions leave ..."

Pp. 88, 89: "Hence, $\bar{O}_1.O_2 v O_3.\bar{S}$ is excluded from verifiability by the suggested condition" *should read*, "Hence, because of the suggested condition, $\bar{O}_1.O_2 v O_3.\bar{S}$ has not been shown to be verifiable".